## Time Reversal in elastodynamics and applications to Structural Health Monitoring

Christos G. Panagiotopoulos, Yiannis Petromichelakis and Chrysoula Tsogka

Institute of Applied \& Computational Mathematics<br>Foundation for Research and Technology Hellas, Heraklion, Greece<br>Department of Mathematics and Applied Mathematics<br>University of Crete, Heraklion, Greece<br>e-mail: tsogka@uoc.gr

COMPDYN 2015

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## Detection and Localization of Damage

- Important in SHM systems
- Response recordings at a number of sensors to monitor structural integrity ${ }^{1}$
- Detection : comparison of recordings to a reference (undamaged) state
- Localization: Inverse Problem usually ill-posed
- Solution: Time-Reversal (TR) computational tool introduced by Fink et. al. ${ }^{2}$
- Achieves refocusing of the wave on the source
- Sending back the recorded signals but reversed in time

1. GE Stavroulakis, (2000) Inverse and crack identification problems in engineering mechanics
2. Fink et. al., (2000) Time-reversed acoustics

## Time Reversal and applications

- TR is a physical process
- It exploits the time reversibility (based on spatial reciprocity and time reversal invariance) of linear wave equations
- Robust and Simple technique for source localization
- Has been applied in Acoustics ${ }^{3}$, Elastodynamics ${ }^{4}$, Electromagnetism, Hydrodynamics etc.
- Finds several applications in medicine, telecommunications, underwater acoustics, seismology, engineering structures, etc.
- Example of source localization in acoustic medium here

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## Time Reversal

- Two step approach :
- Forward step : waves emitted from some source, travel through the medium and the response is recorded by receivers
- Backward step : the recorded signals are time reversed and retransmitted
- Ideal Conditions :
- receivers over the entire domain or it's entire boundary
- recordings of the field variable and its derivatives
- recordings during the whole experiment time $T$
- absence of noise
- Difficulty : the refocusing time is unknown
- Procedure for the assessment of the refocusing time (stopping criterion)


## TR for scatterer localization

- Scatterers act as secondary sources
- Emit at every passage of the original pulse
- Multiple in time sources
- Knowledge of the response in the reference (healthy) configuration
- Scattered field ( $p^{s c a t}=p^{t o t}-p^{r e f}$ ) results better refocusing
- Example of defect localization in acoustic medium

```here
```


## In the present work

- Description of the numerical implementation of TR
- Elastic medium
- Bounded domain $\Omega$
- Excitation produced by $\mathrm{N}_{s}$ point sources forming $\Omega_{s} \subset \Omega$
- Response recordings at $\mathrm{N}_{r}$ receivers forming $\Omega_{r} \subset \Omega$
- $\Omega_{r} \cap \Omega_{s}=\emptyset \quad$ or $\quad \Omega_{r} \cap \Omega_{s} \neq \emptyset \quad$ or $\quad \underline{\Omega_{r}=\Omega_{s}}$
- Sensors may form an array or be distributed
- DORT method ${ }^{5}, 6$ for selective refocusing on multiple defect using the SVD of the Impulse Response Matrix

5. G Derveaux, G Papanicolaou and C Tsogka, (2007) Time reversal imaging for sensor networks with optimal compensation in time
6. E Barbieri and M Meo, (2010) Time reversal DORT method applied to nonlinear elastic wave scattering

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## Forward step

- Simulated Numerically using a mixed finite element formulation ${ }^{7}$
- Wave propagation model

Displacement - Stress (second order)

$$
\begin{aligned}
\rho \frac{\partial^{2} \boldsymbol{u}}{\partial t^{2}}-\operatorname{div} \sigma & =\delta\left(\boldsymbol{x}-\boldsymbol{x}_{s}\right) f(t) \boldsymbol{e}_{i} \\
\sigma & =C: \varepsilon
\end{aligned}
$$

strain - displacement relationship

$$
\varepsilon_{i j}=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial x_{j}}+\frac{\partial u_{j}}{\partial x_{i}}\right)
$$

$$
\dot{\varepsilon}_{i j}=\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right)
$$

- Homogeneous Dirichlet boundary conditions and zero initial conditions

7. E Bécache, P Joly and C Tsogka (2002) A new family of mixed finite elements for the linear elastodynamic problem.

## Backward step

- Always performed numerically in SHM applications
- Three alternative forms
a) imposed displacements at all $\boldsymbol{x}_{r}$
b) appropriate initial conditions
c) sensors acting as sources introducing right hand side loading terms

$$
\begin{aligned}
\rho \frac{\partial \tilde{\boldsymbol{v}}}{\partial t}-\operatorname{div} \tilde{\sigma} & =\sum_{q=1}^{N_{r}} \delta\left(\boldsymbol{x}-\boldsymbol{x}_{q}\right) \boldsymbol{v}\left(\boldsymbol{x}_{q}, T-t\right), & & (\boldsymbol{x}, t) \in \Omega \times(0, T] \\
A: \frac{\partial \tilde{\sigma}}{\partial t}-\dot{\tilde{\varepsilon}} & =0, & & (\boldsymbol{x}, t) \in \Omega \times(0, T] \\
\tilde{\boldsymbol{v}}(\boldsymbol{x}, t) & =0, & & (\boldsymbol{x}, t) \in \partial \Omega \times(0, T] \\
\tilde{\boldsymbol{v}}(\boldsymbol{x}, 0) & =0 & \text { and } & \tilde{\sigma}(\boldsymbol{x}, 0)=0,
\end{aligned}
$$

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## Impulse Response Matrix (IRM)

- In every time step of the forward propagation



## Multiple damaged areas

- DORT method - imaging procedure for selective refocusing on different targets
- Fourier transform of the IRM $P(t)$ to obtain $\hat{P}(\omega)$
- Singular Value Decomposition (SVD) of $\hat{P}(\omega)$ according to

$$
\hat{P}(\omega)=U(\omega) S(\omega) V^{*}(\omega)
$$

- Projection of each column $\hat{P}^{(l)}$ of the transformed IRM on the $k$-th singular vector as

$$
\hat{P}_{k}^{(p)}(\omega)=\left(U_{k}^{*}(\omega) \hat{P}^{(l)}(\omega)\right) V_{k}(\omega)
$$

- Inverse Fourier transform of $\hat{P}_{k}^{(p)}(\omega)$ to obtain $P_{k}^{(p)}(t)$
- Back-propagation of $P_{k}^{(p)}(t)$ to achieve refocusing on one specific defect


## Stopping Criterion

- Refocusing at the defect during the backward propagation
- The refocusing time is not a priori known
- Stopping criteria based on the minimization of some norm of appropriate field quantities
- Shannon entropy
- Bounded variation function of a field variable
- Mathematical energy
- Total energy

$$
\mathcal{E}(t)=\frac{1}{2}(A \sigma, \sigma)+\frac{1}{2}(\rho v, v)
$$

- Definition of the discrete energy density
- Normalization by its maximal value
- Computation of its $L^{1}$ norm


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## Numerical example

- Geometry : rectangular domain $\mathrm{L}_{x}=69.037 \mathrm{~mm}$ and $\mathrm{L}_{y}=62.747 \mathrm{~mm}$
- Mesh : $400 \times 400$ grid with rectangular elements
- Material : steel with Lamé coefficients $\lambda=96.95 \mathrm{GPa}$ and $\mu=76.17 \mathrm{GPa}$
- Velocities : pressure waves $\mathrm{c}_{p}=5689.9 \mathrm{~m} / \mathrm{s}$ and shear waves $\mathrm{c}_{s}=3145.2$ $\mathrm{m} / \mathrm{s}$
- Array of 21 equidistant sensors that act as sources as well
- Two defects : one at $\left(0.6 \mathrm{~L}_{x}, 0.25 \mathrm{~L}_{y}\right)$ of area $0.1 \mathrm{~mm}^{2}$ and second $\left(0.6 \mathrm{~L}_{x}\right.$, $0.75 \mathrm{~L}_{y}$ ) of area $0.4 \mathrm{~mm}^{2}$
- Damage is considered in the material by the degradation of both Lamé coefficients by 10 \%
- Excitation function : Ricker pulse with central frequency 1 MHz


## Results

- Forward step and construction of the IRM
- Back-propagation of the field recorded when the source is on the central array element. Response time-hitory Refocuing
- SVD of the IRM
- Back-propagation of the field recorded when the source is on the central array element after projection on the singular vector corresponding to the
- first singular value

Response time-history

- second singular value

Response time-history

Response time-history

Refocusing
Refocusing

Refocusing

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## Summary and Conclusions

- Time reversal for damage localization in elastic bounded media
- Application of the DORT method for selective refocusing
- Choice of an effective stopping criterion (Total energy)
- Absence of notable differences between array and distributed sensor configurations
- Difficulties in the elastic medium due to the two types of waves (pressure and shear) and their conversions
- Difficulties due to the presence of boundaries


## Future work

- Extensive investigation of the distributed sensor configuration
- Propose optimal total experiment time
- Application of imaging techniques ${ }^{8}$
- Investigation of the methodology using passive noisy recordings as input data
- Account for dissipation (damping) and dispersion
- Application to structures with complex geometry

8. L Borcea, G Papanicolaou, C Tsogka and J Berryman, (2002) Imaging and time reversal in random media

## Source localization

0.471405


## Source localization

1.41421


## Source localization

2.35702


## Source localization

3.29983


## Source localization

4.24264


## Source localization

### 5.18545



## Source localization

### 6.12826



## Source localization

7.07107


## Source localization



## Source localization



## Source localization

9.89949


## Source localization



## Source localization

11.7851


## Source localization

12.7279


## Source localization

13.6707


## Source localization

14.6135


## Source localization



## Source localization

16.4992


## Source localization

17.442


## Source localization



## Source localization

19.3276


## Source localization



## Source localization

21.2132


## Source localization

22.156


## Source localization

23.0988


## Source localization

24.0416


## Source localization

24.9844


## Source localization

25.9272


## Source localization

26.8701


## Source localization



## Source localization

28.7557


## Source localization

29.6985


## Source localization

30.6413


## Source localization

31.5841


## Source localization



## Source localization

33.4697


## Source localization

34.4125


## Source localization



## Source localization

36.2981


## Source localization



## Source localization



## Source localization

39.1266


## Source localization

40.0694


## Source localization

- Time reverse the recordings and rebroadcast hare


## Source localization

0.471405


## Source localization

1.41421


## Source localization

2.35702


## Source localization

3.29983


## Source localization

4.24264


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7.07107


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8.01388


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8.95669


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10.8423


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## Source localization

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31.5841


## Source localization



## Source localization

33.4697


## Source localization

34.4125


## Source localization

35.3553


## Source localization

36.2981


## Source localization



## Source localization

38.1838


## Source localization

39.1266


## Source localization

40.0694


## Defect localization - forward step

Total field 0.333333


Scattered field
0.333333


## Defect localization - forward step

Total field
1


Scattered field


## Defect localization - forward step

Total field
1.66667


Scattered field


## Defect localization - forward step

Total field
2.33333


Scattered field


## Defect localization - forward step

Total field

3


## Scattered field



## Defect localization - forward step

Total field
3.66667


## Scattered field



## Defect localization - forward step

Total field
4.33333


## Scattered field



## Defect localization - forward step

Total field
5


## Scattered field



## Defect localization - forward step

Total field
5.66667


## Scattered field



## Defect localization - forward step

Total field
6.33333


## Scattered field



## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field
7.66667


## Scattered field



## Defect localization - forward step

Total field
8.33333


## Scattered field



## Defect localization - forward step

Total field
9


## Scattered field



## Defect localization - forward step

Total field
9.66667


## Scattered field

9.66667


## Defect localization - forward step

Total field
10.3333


## Scattered field

10.3333


## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field


## Scattered field

11.6667


## Defect localization - forward step

Total field
12.3333


## Scattered field



## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field
13.6667


## Scattered field

13.6667


## Defect localization - forward step

Total field
14.3333


## Scattered field

14.3333


## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field
15.6667


## Scattered field

15.6667


## Defect localization - forward step

Total field
16.3333


## Scattered field

16.3333


## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field
17.6667


## Scattered field

17.6667


## Defect localization - forward step

Total field
18.3333


Scattered field
18.3333


## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field
19.6667


Scattered field
19.6667


## Defect localization - forward step

Total field
20.3333


Scattered field
20.3333


## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field
21.6667


Scattered field
21.6667


## Defect localization - forward step

Total field
22.3333


Scattered field
22.3333


## Defect localization - forward step

Total field

23


Scattered field


## Defect localization - forward step

Total field
23.6667


Scattered field
23.6667


## Defect localization - forward step

Total field
24.3330


Scattered field
24.3333


## Defect localization - forward step

Total field
25


Scattered field


## Defect localization - forward step

Total field


Scattered field
25.6667


## Defect localization - forward step

Total field
26.3333


Scattered field
26.3333


## Defect localization - forward step

Total field


Scattered field


## Defect localization - forward step

Total field
27.6667


Scattered field
27.6667


## Defect localization - forward step

Total field
23.3333


Scattered field
28.3333


## Defect localization - forward step

Total field


## Scattered field



## Defect localization - forward step

Total field
29.6667


Scattered field
29.6667


## Defect localization - forward step

Total field
30.3333


## Scattered field

30.3333


## Defect localization - forward step

Total field


Scattered field


## Defect localization - forward step

Total field


Scattered field
31.6667


- End of forward step
- End of forward step
- Time reverse the recordings and rebroadcast


## Defect localization - backward step

0.666667


## Defect localization - backward step



## Defect localization - backward step

2


## Defect localization - backward step

2.66667


## Defect localization - backward step



## Defect localization - backward step

4


## Defect localization - backward step

4.66667


## Defect localization - backward step



## Defect localization - backward step

6


## Defect localization - backward step

6.66667


## Defect localization - backward step



## Defect localization - backward step

8


## Defect localization - backward step

8.66667


## Defect localization - backward step



## Defect localization - backward step



## Defect localization - backward step

10.6667


## Defect localization - backward step



## Defect localization - backward step



## Defect localization - backward step

12.6667


## Defect localization - backward step



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## Defect localization - backward step

14.6667


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## Defect localization - backward step

16.6667


## Defect localization - backward step



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## Defect localization - backward step

18.6667


## Defect localization - backward step



## Defect localization - backward step



## Defect localization - backward step



## Defect localization - backward step



## Defect localization - backward step



## Defect localization - backward step



## Defect localization - backward step



## Backward step

$$
\mathrm{t}=7.8173 \mathrm{e}-07
$$



## Backward step

## $\mathrm{t}=1.5635 \mathrm{e}-06$



## Backward step

$\mathrm{t}=2.3452 \mathrm{e}-06$


## Backward step

$\mathrm{t}=3.1269 \mathrm{e}-06$


## Backward step

$\mathrm{t}=3.9086 \mathrm{e}-06$


## Backward step

$\mathrm{t}=4.6904 \mathrm{e}-06$


## Backward step

$$
\mathrm{t}=5.4721 \mathrm{e}-06
$$



## Backward step

$\mathrm{t}=6.2538 \mathrm{e}-06$


## Backward step

$\mathrm{t}=7.0355 \mathrm{e}-06$


## Backward step

$$
\mathrm{t}=7.8173 \mathrm{e}-06
$$



## Backward step



## Backward step

$t=9.3807 e-06$


## Backward step

## $\mathrm{t}=1.0162 \mathrm{e}-05$



## Backward step

$$
\mathrm{t}=1.0944 \mathrm{e}-05
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## Backward step

## $\mathrm{t}=1.1726 \mathrm{e}-05$



## Backward step

$$
\mathrm{t}=1.2508 \mathrm{e}-05
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## Backward step

$$
\mathrm{t}=1.3289 \mathrm{e}-05
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## Backward step

$$
\mathrm{t}=1.4071 \mathrm{e}-05
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## Backward step

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\mathrm{t}=1.4853 \mathrm{e}-05
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## Backward step

## $\mathrm{t}=1.5635 \mathrm{e}-05$



## Backward step

$$
\mathrm{t}=1.6416 \mathrm{e}-05
$$



## Backward step

$$
\mathrm{t}=1.7198 \mathrm{e}-05
$$



## Backward step

$$
\mathrm{t}=1.798 \mathrm{e}-05
$$



## Backward step

$$
\mathrm{t}=1.8761 \mathrm{e}-05
$$



## Backward step

$\mathrm{t}=1.9543 \mathrm{e}-05$

$\mathcal{E}(t)=\frac{1}{2}(A \sigma, \sigma)+\frac{1}{2}(\rho v, v)$

$t=1.4853 \mathrm{e}-05$


$$
\mathcal{I}(\mathbf{y})=\int_{0}^{T} \mathcal{E}(\mathbf{y}, t) \mathrm{d} t, \quad \mathbf{y} \in \Omega
$$



## Backward step - first singular value



## Backward step - first singular value

$\mathrm{t}=1.1726 \mathrm{e}-06$


## Backward step - first singular value

$\mathrm{t}=1.9543 \mathrm{e}-06$


## Backward step - first singular value



## Backward step - first singular value

$\mathrm{t}=3.5178 \mathrm{e}-06$


## Backward step - first singular value

$\mathrm{t}=4.2995 \mathrm{e}-06$


## Backward step - first singular value

$\mathrm{t}=5.0812 \mathrm{e}-06$


## Backward step - first singular value

$\mathrm{t}=5.8629 \mathrm{e}-06$


## Backward step - first singular value

$\mathrm{t}=6.6447 \mathrm{e}-06$


## Backward step - first singular value

$t=7.4264 \mathrm{e}-06$


## Backward step - first singular value



## Backward step - first singular value



## Backward step - first singular value

$\mathrm{t}=9.7716 \mathrm{e}-06$


## Backward step - first singular value

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\mathrm{t}=1.0553 \mathrm{e}-05
$$



## Backward step - first singular value

$$
\mathrm{t}=1.1335 \mathrm{e}-05
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## Backward step - first singular value

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\mathrm{t}=1.2117 \mathrm{e}-05
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## Backward step - first singular value

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\mathrm{t}=1.2898 \mathrm{e}-05
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## Backward step - first singular value



## Backward step - first singular value

$$
\mathrm{t}=1.4462 \mathrm{e}-05
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## Backward step - first singular value

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\mathrm{t}=1.5244 \mathrm{e}-05
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## Backward step - first singular value

$\mathrm{t}=1.6025 \mathrm{e}-05$


## Backward step - first singular value



## Backward step - first singular value

$$
\mathrm{t}=1.7589 \mathrm{e}-05
$$



## Backward step - first singular value

$$
\mathrm{t}=1.8371 \mathrm{e}-05
$$



## Backward step - first singular value

$$
\mathrm{t}=1.9152 \mathrm{e}-05
$$



## Backward step - first singular value


$\mathcal{E}(t)=\frac{1}{2}(A \sigma, \sigma)+\frac{1}{2}(\rho v, v)$

$t=1.2898 \mathrm{e}-05$


$$
\mathcal{I}(\mathbf{y})=\int_{0}^{T} \mathcal{E}(\mathbf{y}, t) \mathrm{d} t, \quad \mathbf{y} \in \Omega
$$



## Backward step - second singular value



## Backward step - second singular value

$\mathrm{t}=1.5635 \mathrm{e}-06$


## Backward step - second singular value



## Backward step - second singular value

$\mathrm{t}=3.1269 \mathrm{e}-06$


## Backward step - second singular value

$\mathrm{t}=3.9086 \mathrm{e}-06$


## Backward step - second singular value

$\mathrm{t}=4.6904 \mathrm{e}-06$


## Backward step - second singular value

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\mathrm{t}=5.4721 \mathrm{e}-06
$$



## Backward step - second singular value

$\mathrm{t}=6.2538 \mathrm{e}-06$


## Backward step - second singular value

$\mathrm{t}=7.0355 \mathrm{e}-06$


## Backward step - second singular value

$$
\mathrm{t}=7.8173 \mathrm{e}-06
$$



## Backward step - second singular value



## Backward step - second singular value



## Backward step - second singular value



## Backward step - second singular value

$$
\mathrm{t}=1.0944 \mathrm{e}-05
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## Backward step - second singular value

$$
\mathrm{t}=1.1726 \mathrm{e}-05
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## Backward step - second singular value

$$
\mathrm{t}=1.2508 \mathrm{e}-05
$$



## Backward step - second singular value



## Backward step - second singular value



## Backward step - second singular value

$$
\mathrm{t}=1.4853 \mathrm{e}-05
$$



## Backward step - second singular value

$\mathrm{t}=1.5635 \mathrm{e}-05$


## Backward step - second singular value

## $\mathrm{t}=1.6416 \mathrm{e}-05$



## Backward step - second singular value

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\mathrm{t}=1.7198 \mathrm{e}-05
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## Backward step - second singular value



## Backward step - second singular value

$$
\mathrm{t}=1.8761 \mathrm{e}-05
$$



## Backward step - second singular value

$$
\mathrm{t}=1.9543 \mathrm{e}-05
$$


$\mathcal{E}(t)=\frac{1}{2}(A \sigma, \sigma)+\frac{1}{2}(\rho v, v)$

$t=1.8761 e-05$


$$
\mathcal{I}(\mathbf{y})=\int_{0}^{T} \mathcal{E}(\mathbf{y}, t) \mathrm{d} t, \quad \mathbf{y} \in \Omega
$$



## Backward step - third singular value



## Backward step - third singular value

$\mathrm{t}=1.1726 \mathrm{e}-06$


## Backward step - third singular value

$$
\mathrm{t}=1.9543 \mathrm{e}-06
$$



## Backward step - third singular value



## Backward step - third singular value

$\mathrm{t}=3.5178 \mathrm{e}-06$


## Backward step - third singular value

$\mathrm{t}=4.2995 \mathrm{e}-06$


## Backward step - third singular value

$\mathrm{t}=5.0812 \mathrm{e}-06$


## Backward step - third singular value

$\mathrm{t}=5.8629 \mathrm{e}-06$


## Backward step - third singular value



## Backward step - third singular value

$\mathrm{t}=7.4264 \mathrm{e}-06$


## Backward step - third singular value



## Backward step - third singular value



## Backward step - third singular value

$\mathrm{t}=9.7716 \mathrm{e}-06$


## Backward step - third singular value

$\mathrm{t}=1.0553 \mathrm{e}-05$


## Backward step - third singular value

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\mathrm{t}=1.1335 \mathrm{e}-05
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## Backward step - third singular value

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\mathrm{t}=1.2117 \mathrm{e}-05
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## Backward step - third singular value

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\mathrm{t}=1.2898 \mathrm{e}-05
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## Backward step - third singular value



## Backward step - third singular value



## Backward step - third singular value

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\mathrm{t}=1.5244 \mathrm{e}-05
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## Backward step - third singular value

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\mathrm{t}=1.6025 \mathrm{e}-05
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## Backward step - third singular value

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\mathrm{t}=1.6807 \mathrm{e}-05
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## Backward step - third singular value

$$
\mathrm{t}=1.7589 \mathrm{e}-05
$$



## Backward step - third singular value

$$
\mathrm{t}=1.8371 \mathrm{e}-05
$$



## Backward step - third singular value

$$
\mathrm{t}=1.9152 \mathrm{e}-05
$$



## Backward step - third singular value


$\mathcal{E}(t)=\frac{1}{2}(A \sigma, \sigma)+\frac{1}{2}(\rho v, v)$


$$
\mathcal{I}(\mathbf{y})=\int_{0}^{T} \mathcal{E}(\mathbf{y}, t) \mathrm{d} t, \quad \mathbf{y} \in \Omega
$$

$t=1.2898 \mathrm{e}-05$



[^0]:    3. L Borcea, G Papanicolaou, C Tsogka and J Berryman, (2002) Imaging and time reversal in random media
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